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MORE HASTE, LESS WASTE: LOWERING THE REDUNDANCY IN FULLY INDEXABLE DICTIONARIES

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ABSTRACT. We consider the problem of representing, in a compressed format, a bit-vector S of m bits with n **1**s, supporting the following operations, where $b \in \{\mathbf{0}, \mathbf{1}\}$:

- $\mathtt{rank}_b(S, i)$ returns the number of occurrences of bit b in the prefix $S[1..i]$;
- $\mathtt{select}_b(S, i)$ returns the position of the ith occurrence of bit b in S.

Such a data structure is called *fully indexable dictionary (*FID*)* [Raman, Raman, and Rao, 2007], and is at least as powerful as predecessor data structures. Viewing S as a set $X = \{x_1, x_2, \ldots, x_n\}$ of n distinct integers drawn from a universe $[m] = \{1, \ldots, m\}$, the predecessor of integer $y \in [m]$ in X is given by $\mathtt{select}_1(S, \mathtt{rank}_1(S, y-1))$. FIDs have many applications in succinct and compressed data structures, as they are often involved in the construction of succinct representation for a variety of abstract data types.

Our focus is on space-efficient FIDs on the RAM model with word size $\Theta(\lg m)$ and constant time for all operations, so that the time cost is independent of the input size.

Given the bitstring S to be encoded, having length m and containing n ones, the minimal amount of information that needs to be stored is $B(n,m) = \lceil \log \binom{m}{n} \rceil$. The state of the art in building a FID for S is given in [Pătraşcu, 2008] using $B(m,n) + O(m/((\log m/t)^t)) + O(m^{3/4})$ bits, to support the operations in $O(t)$ time.

Here, we propose a parametric data structure exhibiting a time/space trade-off such that, for any real constants $0 < \delta \leq 1/2$, $0 < \varepsilon \leq 1$, and integer $s > 0$, it uses

$$B(n,m) + O\left(n^{1+\delta} + n\left(\frac{m}{n^s}\right)^{\varepsilon}\right)$$

bits and performs all the operations in time $O(s\delta^{-1} + \varepsilon^{-1})$. The improvement is twofold: our redundancy can be lowered parametrically and, fixing $s = O(1)$, we get a constant-time FID whose space is $B(n,m) + O(m^{\varepsilon}/\mathrm{poly}(n))$ bits, for sufficiently large m. This is a significant improvement compared to the previous bounds for the general case.

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1. Introduction

Data structures for dictionaries [3, 27, 34, 37], text indexing [5, 12, 22, 24, 31, 32], and representing semi-structured data [11, 14, 15, 30, 37], often require the very space-efficient representation of a bivector S of m bits with n $\mathbf{1}$s (and $m-n$ $\mathbf{0}$s). Since there are $\binom{m}{n}$ possible choices of n $\mathbf{1}$s out of the m bits in S, a simple information-theoretic argument shows that we need at least $B(n,m) = \lceil \log \binom{m}{n} \rceil$ bits of space, in the worst case, to store S in some compressed format. However, for the aforementioned applications, it is not enough just to store the compressed S, as one would like to support the following operations on S, for $b \in \{\mathbf{0}, \mathbf{1}\}$:

- $\texttt{rank}_b(S, i)$ returns the number of occurrences of bit b in the prefix $S[1..i]$;
- $\texttt{select}_b(S, i)$ returns the position of the ith occurrence of bit b in S.

Our focus will be on space-efficient data structures that support these operations efficiently, on the RAM model with word size $\Theta(\log m)$. The resulting data structure is called a *fully indexable dictionary* (FID) [37] and is quite powerful. For example, S can equally represent a set $X = \{x_1, x_2, \ldots, x_n\}$ of n distinct integers drawn from a universe $[m] = \{1, \ldots, m\}$, where $S[x_i] = \mathbf{1}$, for $1 \le i \le n$, while the remaining $m - n$ bits of S are $\mathbf{0}$s. In this context, the classical problem of finding the *predecessor* in X of a given integer $y \in [m]$ (i.e. the greatest lower bound of y in X) can be solved with two FID queries on S by $\texttt{select}_1(S, \texttt{rank}_1(S, y-1))$. FIDs have also connections with coding theory, since they represent a sort of locally decodable source code for S [4]. They are at the heart of compressed text indexing since they enable space to be squeezed down to the high-order entropy when properly employed [20]. Finally, they are the building blocks for many complex low space data structures [2, 9, 28, 29] that require $O(1)$ lookup time, namely, their time complexity is independent of the number of entries stored at the expense of using some extra space.

To support the $\texttt{rank}$ and $\texttt{select}$ operations in $O(t)$ time, for some parameter t, it appears to be necessary to use additional space, beyond the bound $B(n,m)$ needed for representing the bitstring S in compressed format. This extra space is termed the *redundancy* $R(n,m,t)$ of the data structure, and gives a total of $B(n,m)+R(n,m,t)$ bits [13]. Although the leading term $B(n,m)$ is optimal from the information-theoretic point of view, a discrepancy between theory and practice emerges when implementing FIDs for various applications [6, 19, 21, 23, 33, 39]. In particular, the term $B(n,m)$ is often of the same order as, if not superseded by, the redundancy term $R(n,m,t)$. For example, consider a constant-time FID storing $n = o(m/\text{polylog}(m))$ integers from the universe $[m]$: here, $B(n,m)$ is negligible when compared to the best known bound of $R(n,m,1) = O(m/\text{polylog}(m))$ [35].

Our goal is that of reducing the redundancy $R(n,m,t)$ for the general case $n \le m$. Although most of the previous work has generally focussed on the case $t = O(1)$, and $m = n \cdot \text{polylog}(n)$, the burgeoning range of applications (and their complexity) warrant a much more thorough study of the function $R(n,m,t)$.

There are some inherent limitations on how small can the redundancy $R(n,m,t)$ be, since FIDs are connected to data structures for the predecessor problem, and we can inherit the predecessor lower bounds regarding several time/space tradeoffs. The connection between FIDs and the predecessor problem is well known [1, 23, 36, 37] and is further developed in this paper, going beyond the simple inheritance of lower bounds. A predecessor data structure which gives access to the underlying data set is, informally, a way to support *half* the operations natively: either $\texttt{select}_1$ and $\texttt{rank}_1$, or $\texttt{select}_0$ and $\texttt{rank}_0$. In fact, we show that a data structure solving the predecessor problem can be turned into a FID and

can also be made to store the data set using $B(n,m)+O(n)$ bits, under certain assumptions over the data structure.

Consequently, if we wish to understand the limitations in reducing the redundancy $R(n,m,t)$ of the space bounds for FIDs, we must briefly survey the state of the art for the lower bounds involving the predecessor problem. The work in [36] shows a number of lower bounds and matching upper bounds for the predecessor problem, using data structures occupying at least $\Omega(n)$ words, from which we obtain, for example, that $R(n,m,1)$ can be $o(n)$ only when $n = \text{polylog}(m)$ (a degenerate case) or $m = n\,\text{polylog}(n)$. For $m = n^{O(1)}$, the lower bound for $B(n,m) + R(n,m,1)$ is $\Omega(n^{1+\delta})$ for any fixed constant $\delta > 0$. Note that in the latter case, $B(n,m) = O(n \log m) = o(R(n,m,1))$, so the "redundancy" is larger than $B(n,m)$. Since $\texttt{rank}_1$ is at least as hard as the predecessor problem, as noted in [1, 36], then all FIDs suffer from the same limitations. (It is obvious that $\texttt{rank}_0$ and $\texttt{rank}_1$ have the same complexity, as $\texttt{rank}_0(S,i) + \texttt{rank}_1(S,i) = i$.) As noted in [37, Lemma 7.3], $\texttt{select}_0$ is also at least as hard as the predecessor problem. Other lower bounds on the redundancy were given for "systematic" encodings of S (see [13, 16, 26] and related papers), but they are not relevant here since our focus is on "non-systematic" encodings [17, 18], which have provably lower redundancy. (In "non-systematic" encodings one can store S in compressed format.)

In terms of upper bounds for $R(n,m,t)$, a number are known, of which we only enumerate the most relevant here. For systematic structures, an optimal upper bound is given by [16] for $R(n,m,O(1)) = O(m \log\log m / \log m)$. Otherwise, a very recent upper bound in [35] gives $R(n,m,t) = O(m/((\log m)/t)^t + m^{3/4}\text{polylog}(m))$ for any constant $t > 0$. These bounds are most interesting when $m = n \cdot \text{polylog}(n)$. As noted earlier, sets that are sparser are worthy of closer study. For such sets, one cannot have best of two worlds: one would either have to look to support queries in non-constant time but smaller space, or give up on attaining $R(n,m,1) = o(B(n,m))$ for constant-time operations.

The main role of generic case FIDs is expressed when they take part in more structured data structures (e.g. succinct trees) where there is no prior knowledge of the relationship between n and m. Our main contribution goes along this path, striving for constant-time operations. Namely, we devise a constant-time FID having redundancy $R(n,m,O(1)) = O(n^{1+\delta} + n(m/n^s)^\varepsilon)$, for any fixed constants $\delta < 1/2$, $\varepsilon < 1$ and $s > 0$ (Theorem 3.1). The running time of the operations is always $O(1)$ for $\texttt{select}_1$ (which is insensitive to time-space tradeoffs) and is $O(\varepsilon^{-1} + s\delta^{-1}) = O(1)$ for the remaining operations. When m is sufficiently large, our constant-time FID uses just $B(n,m) + O(m^\varepsilon/\text{poly}(n))$ bits, which is a significant improvement compared to the previous bounds for the general case, as we move from a redundancy of kind $O(m/\text{polylog}(m))$ to a one of kind $O(m^\varepsilon)$, by proving for the first time that polynomial reduction in space is possible.

Moreover, when instantiated in a polynomial universe case (when $m = \Theta(n^{O(1)})$, for a sufficiently small ε, the redundancy is dominated by $n^{1+\delta}$, thus extending the known predecessor search data structure with all four FID operations without using a second copy of the data. Otherwise, the m^ε term is dominant when the universe is superpolynomial, e.g. when $m = \Theta(2^{\log^c n})$ for $c > 1$. In such cases we may not match the lower bounds for predecessor search; however, this is the price for a solution which is agnostic of m, n relationship.

We base our findings on the Elias-Fano encoding scheme [7, 8], which gives the basis for FIDs naturally supporting $\texttt{select}_1$ in $O(1)$ time.

2. Elias-Fano Revisited

We review how the Elias-Fano scheme [7, 8, 33, 39] works for an arbitrary set $X = \{x_1 < \cdots < x_n\}$ of n integers chosen from a universe $[m]$. Recall that X is equivalent to its characteristic function mapped to a bitstring S of length m, so that $S[x_i] = \mathbf{1}$ for $1 \leq i \leq n$ while the remaining $m-n$ bits of S are $\mathbf{0}$s. Based on the Elias-Fano encoding, we will describe the main ideas behind our new implementation of fully indexable dictionaries (FIDs). We also assume that $n \leq m/2$—otherwise we build a FID on the complement set of X (and still provide the same functionalities), which improves space consumption although it does not guarantee $\texttt{select}_1$ in $O(1)$ time.

Elias-Fano encoding. Let us arrange the integers of X as a *sorted* sequence of consecutive words of $\log m$ bits each. Consider the first[1] $\lceil \log n \rceil$ bits of each integer x_i, called h_i, where $1 \leq i \leq n$. We say that any two integers x_i and x_j belong to the same *superblock* if $h_i = h_j$.

The sequence $h_1 \leq h_2 \leq \cdots \leq h_n$ can be stored as a bitvector H in $3n$ bits, instead of using the standard $n\lceil \log n \rceil$ bits. It is the classical unary representation, in which an integer $x \geq 0$ is represented with x $\mathbf{0}$s followed by a $\mathbf{1}$. Namely, the values $h_1, h_2 - h_1, \ldots, h_n - h_{n-1}$ are stored in unary as a multiset. For example, the sequence $h_1, h_2, h_3, h_4, h_5 = 1, 1, 2, 3, 3$ is stored as $H = \mathbf{01101011}$, where the ith $\mathbf{1}$ in H corresponds to h_i, and the number of $\mathbf{0}$s from the beginning of H up to the ith $\mathbf{1}$ gives h_i itself. The remaining portion of the original sequence, that is, the last $\log m - \lceil \log n \rceil$ bits in x_i that are not in h_i, are stored as the ith entry of a simple array L. Hence, we can reconstruct x_i as the concatenation of h_i and $L[i]$, for $1 \leq i \leq n$. The total space used by H is at most $2^{\lceil \log n \rceil} + n \leq 3n$ bits and that used by L is $n \times (\log m - \lceil \log n \rceil) \leq n \log(m/n)$ bits.

Interestingly, the plain storage of the bits in L is related to the information-theoretic minimum, namely, $n \log(m/n) \leq B(n,m)$ bits, since for $n \leq m/2$, $B(n,m) \sim n \log(m/n) + 1.44\,n$ by means of Stirling approximation. In other words, the simple way of representing the integers in X using Elias-Fano encoding requires at most $n \log(m/n) + 3n$ bits, which is nearly $1.56\,n$ away from the theoretical lower bound $B(n,m)$. If we employ a constant-time FID to store H, Elias-Fano encoding uses a total of $B(n,m) + 1.56\,n + o(n)$ bits.

Rank and select operations vs predecessor search. Using the available machinery—the FID on H and the plain array L—we can perform $\texttt{select}_1(i)$ on X in $O(1)$ time: we first recover $h_i = \texttt{select}_1(H, i) - i$ and then concatenate it to the fixed-length $L[i]$ to obtain x_i in $O(1)$ time [22]. As for $\texttt{rank}$ and $\texttt{select}_0$, we point out that they are intimately related to the *predecessor search*, as we show below (the converse has already been pointed out in the Introduction).

Answering $\texttt{rank}_1(k)$ in X is equivalent to finding the predecessor x_i of k in X, since $\texttt{rank}_1(k) = i$ when x_i is the predecessor of k. Note that $\texttt{rank}_0(k) = k - \texttt{rank}_1(k)$, so performing this operation also amounts to finding the predecessor. As for $\texttt{select}_0(i)$ in X, let $\overline{X} = [m] \setminus X = \{v_1, v_2, \ldots, v_{m-n}\}$ be the complement of X, where $v_i < v_{i+1}$, for $1 \leq i < m-n$. Given any $1 \leq i \leq m-n$, our goal is to find $\texttt{select}_0(i) = v_i$ in constant time, thus motivating that our assumption $n \leq m/2$ is w.l.o.g.: whenever $n \leq m/2$, we store the complement set of X and swap the zero- and one-related operations.

The key observation comes from the fact that we can associate each x_l with a new value $y_l = \left|\{v_j \in \overline{X} \text{ such that } v_j < x_l\}\right|$, which is the number of elements in $\overline{X}$ that precede x_l,

[1] Here we use Elias' original choice of ceiling and floors, thus our bounds slightly differ from the *sdarray* structure of [33], where they obtain $n\lceil \log(m/n) \rceil + 2n$.

where $1 \leq l \leq n$. The relation among the two quantities is simple, namely, $y_l = x_l - l$, as we know that exactly $l-1$ elements of X precede x_l and so the remaining elements that precede x_l must originate from $\overline{X}$. Since we will often refer to it, we call the set $Y = \{y_1, y_2, \ldots, y_n\}$ the *dual representation* of the set X.

Returning to the main problem of answering $\texttt{select}_0(i)$ in X, our first step is to find the predecessor y_j of i in Y, namely, the largest index j such that $y_j < i$. As a result, we infer that x_j is the predecessor of the *unknown* v_i (which will be our answer) in the set X. We now have all the ingredients to deduce the value of v_i. Specifically, the y_jth element of $\overline{X}$ occurs before x_j in the universe, and there is a nonempty run of elements of X up to and including position x_j, followed by $i - y_j$ elements of $\overline{X}$ up to and including (the unknown) v_i. Hence, $v_i = x_j + i - y_j$ and, since $y_j = x_j - j$, we return $v_i = x_j + i - x_j + j = i + j$. (An alternative way to see $v_i = i + j$ is that $x_1, x_2, \ldots, x_j$ are the only elements of X to the left of the unknown v_i.) We have thus proved the following.

Lemma 2.1. *Using the Elias-Fano encoding, the* $\texttt{select}_1$ *operation takes constant time, while the* $\texttt{rank}$ *and* $\texttt{select}_0$ *operations can be reduced in constant time to predecessor search in the sets* X *and* Y*, respectively.*

The following theorem implies that we can use both lower and upper bounds of the predecessor problem to obtain a FID, and vice versa. Below, we call a data structure storing X *set-preserving* if it stores $x_1, \ldots, x_n$ *verbatim* in a contiguous set of memory cells.

Theorem 2.2. *For a given set* X *of* n *integers over the universe* $[m]$*, let* $\text{FID}(t, s)$ *be a* FID *that takes* t *time and* s *bits of space to support* $\texttt{rank}$ *and* $\texttt{select}$*. Also, let* $\text{PRED}(t, s)$ *be a static data structure that takes* t *time and* s *bits of space to support predecessor queries on* X*, where the integers in* X *are stored in sorted order using* $n \log m \leq s$ *bits. Then,*

(1) *given a* $\text{FID}(t, s)$*, we can obtain a* $\text{PRED}(O(t), s)$*;*
(2) *given a set-preserving* $\text{PRED}(t, s)$*, we can obtain a* $\text{FID}(O(t), s - n \log n + O(n))$ *(equivalently,* $R(n, m, t) = s - n \log m + O(n)$*) with constant-time* $\texttt{select}_1$*.*
(3) *if there exists a non set-preserving* $\text{PRED}(t, s)$*, we can obtain a* $\text{FID}(O(t), 2s + O(n))$ *with constant-time* $\texttt{select}_1$*.*

Proof (sketch). The first statement easily follows by observing that the predecessor of k in X is returned in $O(1)$ time by $\texttt{select}_1(S, \texttt{rank}_1(S, k-1))$, where S is the characteristic bitstring of X. Focusing on the second statement, it suffices to encode X using the Elias Fano encoding, achieving space $s - n \log n + O(n)$.

To further support $\texttt{select}_0$, we exploit the properties of Y and X. Namely, there exists a maximal subset $X' \subseteq X$ so that its dual representation Y' is strictly increasing, thus being searchable by a predecessor data structure. Hence we split X into X' and the remaining subsequence X'' and produce two Elias-Fano encodings which can be easily combined by means of an extra $O(n)$ bits FID in order to perform $\texttt{select}_1$, $\texttt{rank}_1$ and $\texttt{rank}_0$. $\texttt{select}_0$ can be supported by exploiting the set preserviness of the data structure, thus building only the extra data structure to search Y' and not storing Y'. When data structures are not set-preserving, we simply replicate the data and store Y', thus giving a justification to the $O()$ factor.

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3. Basic Components and Main Result

We now address and solve two questions, which are fundamental to attain a $O(t)$-time FID with $B(n,m)+R(n,m,t)$ bits of storage using Lemma 2.1 and Theorem 2.2: (1) how to devise an efficient index data structure that can implement predecessor search using Elias-Fano representation with tunable time-space tradeoff, and (2) how to keep its redundancy $R(n,m,t)$ small.

Before answering the above questions, we give an overview of the two basic tools that are adopted in our construction (the string B-tree [10] and a modified van Emde Boas tree [36, 38]). We next develop our major ideas that, combined with these tools, achieve the desired time-space tradeoff, proving our main result.

Theorem 3.1. *Let $s > 0$ be an integer and let $0 \le \varepsilon, \delta \le 1$ be reals. For any bitstring S, $|S| = m$, having cardinality n, there exists a fully indexable dictionary solving all operations in time $O(s\delta^{-1} + \varepsilon^{-1})$ using $B(n,m) + O(n^{1+\delta} + n(m/n^s)^{\varepsilon})$ bits of space.*

Modified van Emde Boas trees. Pătraşcu and Thorup [36] have given some matching upper and lower bounds for the predecessor problem. The discussion hereafter regards the second branch of their bound: as a candidate bound they involve the equation (with our terminolgy and assuming our word RAM model) $t = \log(\log(m/n)/\log(z/n))$, where t is our desired time bound and z is the space in bits. By reversing the equation and setting $\epsilon = 2^{-t}$, we obtain $z = \Theta(n(m/n)^{\epsilon})$ bits. As mentioned in [36], the tradeoff is tight for a polynomial universe $m = n^{\gamma}$, for $\gamma > 1$, so the above redundancy cannot be lower than $\Theta(n^{1+\delta})$ for any fixed $\delta > 0$.

They also describe a variation of van Emde Boas (VEB) trees [38] matching the bound for polynomial universes, namely producing a data structure supporting predecessor search that takes $O(\log \frac{\log(m/n)}{\log(z/n)})$ time occupying $O(z \log m)$ bits. In other words, for constant-time queries, we should have $\log(m/n) \sim \log(z/n)$, which implies that the space is $z = \Theta(n(m/n)^{\epsilon})$. They target the use of their data structure for polynomial universes, since for different cases they build different data structures. However, the construction makes no assumption on the above relation and we can extend the result to arbitrary values of m. By Theorem 2.2, we can derive a constant-time FID with redundancy $R(n,m,O(1)) = O(n(m/n)^{\epsilon})$.

Corollary 3.2. *Using a modified VEB tree, we can implement a FID that uses $B(n,m) + O(n(m/n)^{\varepsilon})$ bits of space, and supports all operations in $O(\log(1/\varepsilon))$ time, for any constant $\varepsilon > 0$.*

The above corollary implies that we can obtain a first polynomial reduction by a straightforward application of existing results. However, we will show that we can do better for sufficiently large m, and effectively reduce the term $n(m/n)^{\varepsilon}$ to $n^{1+\delta} + n(m/n^s)^{\varepsilon}$. The rest of the paper is devoted to this goal.

String B-Tree: blind search for the integers. We introduce a variant of string B-tree to support predecessor search in a set of integers. Given a set of integers $X = \{x_1, \ldots, x_p\}$ from the universe $[u]$, we want obtain a space-efficient representation of X that supports predecessor queries efficiently. We develop the following structure:

Lemma 3.3. *Given a set X of p integers from the universe $[u]$, there exists a representation that uses extra $O(p \log\log u)$ bits apart from storing the elements of X, that supports*

predecessor queries on X in $O(\log p/\log\log u)$ time. The algorithm requires access to a precomputed table of size $O(u^\gamma)$ bits, for some positive constant $\gamma < 1$, which can be shared among all instances of the structure with the same universe size.

Proof. The structure is essentially a succinct version of string B-tree on the elements of X interpreted as binary strings of length $\log u$, with branching factor $b = O(\sqrt{\log u})$. Thus, it is enough to describe how to support predecessor queries in a set of b elements in constant time, and the query time follows, as the height of the tree is $O(\log p/\log\log u)$. Given a set $x_1, x_2, \ldots, x_b$ of integers from $[u]$ that need to be stored at a node of the string B-tree, we construct a compact trie (Patricia trie) over these integers (interpreted as binary strings of length $\log u$), having b leaves and $b-1$ internal nodes. The leaves disposition follows the sorting order of X. Each internal node is associated with a *skip value*, indicating the string depth at which the LCP with previous string ends. Canonically, left-pointing edges are labeled with a **0** and right-pointing with a **1**. Apart from storing the keys in sorted order, it is enough to store the tree structure and the skip values of the edges. This information can be represented using $O(b\log\log u)$ bits, as each skip value is at most $\log u$ and the trie is represented in $O(b)$ bits.

Given an element $y \in [u]$, the search for the predecessor of y proceeds in two stages. In the first stage, we simply follow the compact trie matching the appropriate bits of y to find a leaf v. Let x_i be the element associated with leaf v. One can show that x_i is the key that shares the longest common prefix with y among all the keys in X. In the second stage, we compare y with x_i to find the longest common prefix of y and x_i (which is either the leftmost or rightmost leaf of the internal node at which the search ends). By following the path in the compact trie governed by this longest common prefix, one can find the predecessor of y in X. We refer the reader to [10] for more details and the correctness of the search algorithm. The first stage of the search does not need to look at any of the elements associated with the leaves. Thus this step can be performed using a precomputed table of size $O(u^\gamma)$ bits, for some positive constant $\gamma < 1$ (by dividing the binary representation of y into chunks of size smaller than $\gamma\log u$ bits each). In the second stage, finding the longest common prefix of y and x_i can be done using bitwise operations. We again use the precomputed table to follow the path governed by the longest common prefix, to find the predecessor of y. ∎

4. Main Ideas for Achieving Polynomial Redundancy

In this section, we give a full explanation of the main result, Theorem 3.1. We first give an overview, and then detail the multiranking problem by illustrating remaining details involving the construction of our data structure.

4.1. Overview of our recursive dictionary

We consider the $\mathtt{rank}_1$ operation only, leaving the effective development of the details to the next sections. A widely used approach to the FID problem (e.g. see [25, 27]) lies in splitting the universe $[m]$ into different chunks and operating independently in each chunk, storing the rank at the beginning of the block. Queries are redirected into a chunk via a preliminary *distributing* data structure and the *local* data structure is used to solve it. Thus, the space occupancy is the distributing structure (once) plus all chunks. Our approach is

orthogonal, and it guarantees better control of the parameter of subproblems we instantiate with respect to many previous approaches.

Let X ($|X| = n$) be the integer sequence of values drawn from $[m]$ and let $q \in [m]$ be a generic rank query. Our goal is to produce a simple function $f : [m] \to [m/n]$ and a machinery that generates a sequence $\tilde{X}$ from X of length n coming from the universe $[m/n]$, so that given the predecessor of $\tilde{q} = f(q)$ in $\tilde{X}$, we can recover the predecessor of q in X. By this way, we can reduce recursively, multiple times, the rank problem while keeping a single sequence per step, instead of having one data structure per chunk.

Easily enough, f is the "cutting" operation of the upper $\log n$ bits operated by the Elias Fano construction, which generates p different superblocks. Let $X_1^l, \ldots, X_p^l$ the sets of lower $\log(m/n)$ bits of values in X, one per superblock. We define our $\tilde{X}$ as $\tilde{X} = \cup_{1 \le i \le p} X_i^l$, that is, the set of unique values we can extract from the X^ls. Suppose we have an oracle function ψ, so that given a value $\tilde{x} \in \tilde{X}$ and an index $j \in [p]$, $\psi(j, \tilde{x})$ is the predecessor of $\tilde{x}$ in X_j^l. We also recall from Section 2 that the upper bit vector H of the Elias Fano construction over X can answer the query $\mathtt{rank}_1(x/2^{\lceil \log n \rceil})$ in constant time (by performing $\mathtt{select}_0(H, x/2^{\lceil \log n \rceil})$. That is, it can give the rank value at the beginning of each superblock.

Given a query q we can perform $\mathtt{rank}_1(q)$ in the following way: we use H to reduce the problem within the superblock and know the rank at the beginning of the superblock j. We then have the lower bits of our query ($f(q)$) and the sequence $\tilde{X}$: we rank $f(q)$ there, obtaining a certain result, say v; we finally refer to our oracle to find the predecessor of v into X_j^l, and thus find the real answer for $\mathtt{rank}_1(q)$. The main justification of this architecture is the following: in any superblock, the predecessor of some value can exhibit only certain values in its lower bits (those in $\tilde{X}$), thus once given the predecessor of $f(q)$ our necessary step is only to reduce the problem within $[|\tilde{X}|]$ as the lower bits for any superblock are a subset of $\tilde{X}$. The impact of such choice is, as explained later, to let us implement the above oracle in just $O(n^{1+\delta})$ bits, for any $0 < \delta < 1$. That is, by using a superlinear number of bits in n, we will be able to let m drop polynomially both in n and m.

The above construction, thus, requires one to write X in an Elias Fano dictionary, plus the oracle space and the space to solve the predecessor problem on $\tilde{X}$. The first part accounts for $B(n, m) + O(n)$ bits, to which we add $O(n^{1+\delta})$ bits for the oracle. By carefully employing the String B-tree we can shrink the number of elements of $\tilde{X}$ to $O(n/\log^2 n)$ elements, leaving us with the problem of ranking on a sequence of such length and universe $[m/n]$. We solve the problem by replicating the entire schema from the beginning. Up to the final stage of recursion, the series representing the space occupancy gives approximately $O((n \log(m/n))/\log^{2i} n + (n/\log^{2i} n)^{1+\delta})$ bits at the i-th step, descending geometrically. Each step can be traversed in constant time during a query, so the overall time is constant again. More interestingly, at each step we reduce the universe size of the outcoming sequence to mn^{-i}. Thus, at the final step s, we employ the previous result of Corollary 3.2 and obtain a final redundancy of $O(m^{\varepsilon} n^{1-s\varepsilon})$.

4.2. Multiranking

We now give further details on our construction. Mainly, we show that using our choice on how to build $\tilde{X}$ and the function f, being able to rank over $\tilde{X}$ we can build the oracle in $O(n^{1+\delta})$ bits. We do it by illustrating, in a broader framework, the multiranking problem.

We are given a *universe* $[u]$ (in our dictionary case, we start by setting $u = m$), and a set of nonempty sequences $A_1, \ldots, A_c$ each containing a sorted subset of $[u]$. We also define $r = \sum_{1 \le j \le n} |A_j|$ as the global number of elements. The goal is, given two values $1 \le i \le c$ (the wanted superblock $\hat{s}$) and $1 \le q \le u$ (the query $f(q)$), perform $\mathtt{rank}_1(q)$ in the set A_i (in our case, the head in $\hat{s}$ that is predecessor of the searched key) in $O(1)$ time and small space.

A trivial solution to this problem would essentially build a FID for each of the sequences, thus spending a space proportional to $O(cu)$, which is prohibitive. Instead, we can carefully exploit the global nature of this task and solve it in less space. The core of this technique is the *universe scaling* procedure. We perform the union of all the A sequences and extract a new, single sequence Λ containing only the distinct values that appear in the union (that is, we kill duplicates). Λ is named the *alphabet* for our problem and we denote its length with $t \le r$. Next, we rewrite all sequences by using rank of their elements in the alphabet instead of the initial arguments: now each sequence is defined on $[t]$.

The multiranking problem is solved in two phases. We first perform ranking of the query q on Λ and then we exploit the information to recover the predecessor in the given set. Here we achieve our goal to *(i)* decouple a phase that depends on the universe from one that dependes on the elements and *(ii)* have only one version of the problem standing on the initial universe. The following lemma solves the multiranking problem completely, that is, outside our original distinction between a oracle and the alphabet ranking:

Lemma 4.1. *There exists a data structure solving the* `multirank` *problem over c nonempty increasing sequences* $\mathbb{A} = \{A_1, \ldots, A_c\}$ *with elements drawn from the universe* $[u]$*, having r elements in total using* $B(r, u) + O(r^{1+\delta}) + o(u)$ *bits for any given* $0 < \delta < 1/2$.

Proof. Let Λ be the alphabet defined over u by the sequences in $\mathbb{A}$, and let $t = |\Lambda|$. For each of the sequences in $\mathbb{A}$ we create a bitvector β_i of length t where the $\beta_{ij} = \mathbf{1}$ if $\Lambda_j \in A_i$. We first view β_is as rows of a matrix of size tc; since $t \le r$ and each of the sequences are non-empty (and hence $r \ge c$), the matrix is of size $O(r^2)$. We linearize the matrix by concatenating its rows and obtain a new bitvector β' on which we want to perform predecessor search. We note that the universe size of this bitvector is $O(r^2)$, that is, the universe is polynomial. We store β' using the data structure of Corollary 3.2 setting the time to $\log(1/\delta)$, so that space turns out to be $O(r^{1+\delta})$. Finally, we store we store a FID occupying $B(r, u) + o(u)$ that represents the subset Λ of the universe $[u]$.

Solving the multirank is easy now: given a query q and a set index i, we use the $o(u)$ FID and find $\lambda = \mathtt{rank}_1(q)$ in U, which leads to the predecessor into the alphabet Λ of our query q. Since $\lambda \in [t]$ we can now use the β FID to find $p = \mathtt{rank}_1(ti + \lambda)$. The final answer is clearly $p - \mathtt{rank}_1(ti)$. ∎

4.3. Completing the puzzle

The multiranking problem is closely connected with the Elias-Fano representation of Section 2. When plugged in our framework, as explained in Section 4.1, that we can use our data structure itself to implement the ranking procedure. Similarly we can use it for $\mathtt{select}_0$ by employing another set of data.

We are left with just one major detail. Each time we produce the output sequence $\tilde{X}$, containing the lower bits for all elements, our only clue for the number of elements is the worst case upper bound n, which is unacceptable. We now review the whole construction

and employ the string B-tree to have a polylogarithmic reduction on the number of elements, paying $O(n \log\log m)$ bits per recursion step. Generally, at each step we receive a sequence X_i as input and must output a new sequence X_{i+1} plus some data structures that can link the predecessor problem for X_i to X_{i+1}. Each X_i is stored in an Elias-Fano dictionary, and the sets of superblocks and lower bits sequences are built as explained before. We then apply a further reduction step on the problem cardinality. Each superblock can be either *slim* or *fat* depending on whether it contains less than $\log^2 n$ elements or not. Each superblock is split into blocks of size $\log^2 n$, apart from the last block, and for each block we store a String B-tree with fan-out $\sqrt{\log n}$. Since the block is polylogarithmic in size, by means of shared precomputed tables we can perform predecessor search in constant time. Slim superblocks are handled directly by the tree and they do not participate further in the construction. For each block in a fat superblock, we logically extract its *head*, that is, the smallest element in it. We now use heads in the multiranking problems and we build the output sequence X_{i+1} using only heads lower bits. As there can only be at most $O(n/\log^2 n)$ blocks in fat superblocks, the size of the output sequence is at most $O(n/\log^2 n)$. The oracle is built as usual, on the heads, using $O(n^{1+\delta})$ bits.

Ranking now performs the following steps: for each recursive step, it uses the Elias-Fano H vector to move into a superblock and at the same time check if it is slim or fat. In the latter case, it first outsources the query for the lower bits to the next dictionary, then feeds the answer to the multiranking instance and returns the actual answer. Thus, we just proved the following (with $v = \log^2 n$ and $w = n$):

Theorem 4.2. *Let w and v be two integer parameters and let $0 < \delta < 1/2$ be a real constant. Given $X_i, n_i \geq v$ and $m_i > w$, where $n_i \leq m_i$, there exists a procedure that produces a data structure involved in predecessor search. The data structure occupies $B(n_i, m_i) + O(w + n_i \log\log m_i + n_i^{1+\delta})$ space, and in $O(\delta^{-1})$ time, it reduces a predecessor query on X_i to a predecessor query on a new sequence X_{i+1} of length $n_{i+1} = O(n_i/v)$ over a universe $[m_{i+1}]$, where $m_{i+1} = m_i/w$.*

We must then deal with the last two steps. The first step aims at supporting $\mathtt{select}_0$ since the above data structure can only support $\mathtt{rank}_1$. The second step deals with how treat the final sequence after a number of iteration steps have been executed. We can finally give the proof of our main result:

Proof of Theorem 3.1. Let $X \subseteq [m]$ be the set whose characteristic vector is S. The data structure involves recursive instances of Theorem 4.2, by starting with $X_0 = X$ and using each step's output as input for the next step. As previously mentioned, we must only cover the base case and the last recursive step. We begin by describing the whole data structure, moving to algorithms later on. We start by partitioning X into X' and X'' as described in the proof of Theorem 2.2, so that the construction is operated on both X' and X''. We now describe representation of X'; X'' is stored in a similar way. We recursively build smaller sequences by invoking Theorem 4.2 exactly s times, using δ as given, and parameters $w = n$, $v = \log^2 m$. By invoking Corollary 3.2 the space bound easily follows. To support $\mathtt{select}_0$ on the original sequence, we operate on the X' sequence alone, since when transformed to its dual Y', we obtain a strictly monotone sequence. Interpreting X' as an implicit representation of Y' we build a multiset representation for the high bits (H'), a new set of succinct string B-trees using the superblocks of the dual sequence and thought of as operating on Y' (similarly to Theorem 2.2) and a new set of s recursive applications of Theorem 4.2.

$\mathtt{select}_1$ is trivial, thanks to the machinery of Theorem 2.2. The $\mathtt{rank}_1$ algorithm for a query q is performed on both X' and X'' FID: we start by querying H_0, the upper bits of F_0' (F_0'' respectively) for $q/2^{\lceil \log n \rceil}$, thus identifying a certain superblock in which the predecessor for q can appear. Unless the superblock is slim (refer to proof of Theorem 4.2) we must continue to search through the next lower-order bits. This is done via multiranking, which recurses in a cascading manner with the same technique on the s steps up to the last FID, that returns the answer. The chain is then walked backwards to find the root FID representative. We finally proceed through the succinct string B-tree to find the head and the next succinct string B-tree until we find the predecessor of q. The last step for recursion takes $O(\varepsilon^{-1})$ time. All the middle steps for multiranking and succinct string B-tree traversals take $O(s\delta^{-1} + s)$ time. To support $\mathtt{select}_0$, we act on X', using exactly the same algorithm as before using, but with the collection of data structures built for the dual representation Y', and following the steps of Theorem 2.2.

During the buildup of the recursive process, say being at step i, the size n_i' for sequence X_i' ($i > 1$), is upper bounded by $n/\log^{2i} m$, while the universe has size m/n^i. If at any step $2 \le j \le s$ the condition $m_j < w = n$ does not apply, we cannot apply Theorem 4.2, so we truncate recursion and use a $o(w)$ FID to store the sequence X_j. This contributes a negligible amount to the redundancy. We name the FID for each step F_1 up to F_s. Suppose we can recurse for s steps with Theorem 4.2, we end up with a sequence over a universe $m_s = m/n^s$. By using Corollary 3.2 the space bound is no less than $O(n(m/n^s)^\varepsilon)$. The $B(n_i, m_i) + O(n_i^{1+\delta})$ factors decrease geometrically, so the root dominates and we can show that, apart from lower order terms, the space bound is as claimed. Otherwise, the total space $s(n_i, m_i)$ of the recursive data structure satisfies:

$$s(n_i, m_i) = s(n_{i+1}, m_{i+1}) + \text{space}(\text{FID for high bits}) + \text{space}(\text{string B-trees}) + O(n_i^{1+\delta})$$

where $n_{i+1} = n_i/\log^2 m$ and $m_{i+1} = m_i/n$. The claimed redundancy follows easily. ∎

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References

[1] P. Beame and F.E. Fich. Optimal bounds for the predecessor problem and related problems. *J. Comput. Syst. Sci*, 65:38–72, 2002.

[2] D. K. Blandford and G. E. Blelloch. Compact dictionaries for variable-length keys and data with applications. *ACM Transactions on Algorithms*, 4(2):17:1–17:25, May 2008.

[3] A. Brodnik and J.I. Munro. Membership in constant time and almost-minimum space. *SIAM J. Computing*, 28:1627–1640, 1999.

[4] H. Buhrman, P. B. Miltersen, J. Radhakrishnan, and S. Venkatesh. Are bitvectors optimal? *SIAM Journal on Computing*, 31(6):1723–1744, December 2002.

[5] D.R. Clark and J.I. Munro. Efficient suffix trees on secondary storage. In *Proc. 7th ACM-SIAM SODA*, pages 383–391, 1996.

[6] F. Claude and G. Navarro. Practical rank/select queries over arbitrary sequences. In *Proc. 15th (SPIRE)*, LNCS. Springer, 2008.

[7] P. Elias. Efficient storage and retrieval by content and address of static files. *J. Assoc. Comput. Mach.*, 21(2):246–260, 1974.

[8] R. M. Fano. On the number of bits required to implement an associative memory. *Memorandum 61, Computer Structures Group, Project MAC*, 1971.

[9] P. Ferragina, R. Grossi, A. Gupta, R. Shah, and J. S. Vitter. On searching compressed string collections cache-obliviously. In *Proc. 27th ACM PODS*, pages 181–190, 2008.
[10] P. Ferragina and Roberto Grossi. The string b-tree: A new data structure for string search in external memory and its applications. *J. of the ACM*, 46(2):236–280, 1999.
[11] P. Ferragina, F. Luccio, G. Manzini, and S. Muthukrishnan. Structuring labeled trees for optimal succinctness, and beyond. In *Proc. 46th IEEE FOCS*, pages 184–196, 2005.
[12] P. Ferragina and G. Manzini. Indexing compressed text. *Journal of the ACM*, 52(4):552–581, July 2005.
[13] A. Gál and P. Bro Miltersen. The cell probe complexity of succinct data structures. *Theor. Comput. Sci.*, 379:405–417, 2007.
[14] R. F. Geary, N. Rahman, R. Raman, and V. Raman. A simple optimal representation for balanced parentheses. *Theor. Comput. Sci.*, 368:231–246, 2006.
[15] R. F. Geary, R. Raman, and V. Raman. Succinct ordinal trees with level-ancestor queries. *ACM Transactions on Algorithms*, 2:510–534, 2006.
[16] A. Golynski. Optimal lower bounds for rank and select indexes. *Theor. Comput. Sci.*, 387:348–359, 2007.
[17] A. Golynski, R. Grossi, A. Gupta, R. Raman, and S. S. Rao. On the size of succinct indices. In *Proc 15th ESA, LNCS 4698*, pages 371–382, 2007.
[18] A. Golynski, R. Raman, and S. S. Rao. On the redundancy of succinct indices. In *Proc. 11th SWAT*, pages 148–159, 2008.
[19] R. González, Sz. Grabowski, V. Mäkinen, and G. Navarro. Practical implementation of rank and select queries. In *Proc. 4th (WEA)*, pages 27–38, 2005.
[20] R. Grossi, A. Gupta, and J. S. Vitter. High-order entropy-compressed text indexes. In *Proc. 14th ACM-SIAM SODA*, pages 841–850, 2003.
[21] R. Grossi, A. Gupta, and J. S. Vitter. When indexing equals compression: experiments with compressing suffix arrays and applications. In *Proc. 15th ACM-SIAM SODA*, pages 636–645, 2004.
[22] R. Grossi and J. S. Vitter. Compressed suffix arrays and suffix trees with applications to text indexing and string matching. *SIAM J. Comput*, 35(2):378–407, 2005.
[23] A. Gupta, W. Hon, R. Shah, and Jeffrey Scott Vitter. Compressed data structures: Dictionaries and data-aware measures. *Theor. Comput. Sci*, 387(3):313–331, 2007.
[24] W. Hon, K. Sadakane, and W. Sung. Breaking a time-and-space barrier in constructing full-text indices. In *Proc. 44th IEEE FOCS*, pages 251–260, 2003.
[25] G. Jacobson. *Succinct Static Data Structures*. PhD thesis, Carnegie Mellon University, 1989.
[26] P. B. Miltersen. Lower bounds on the size of selection and rank indexes. In *Proc. ACM-SIAM SODA*, pages 11–12, 2005.
[27] J. I. Munro. Tables. In *Proc. FST & TCS, LNCS 1180*, pages 37–42, 1996.
[28] J. I. Munro. Lower bounds for succinct data structures. In *Proc. 19th CPM*, page 3, 2008.
[29] J I. Munro, R. Raman, V. Raman, and S. S. Rao. Succinct representations of permutations. In *Proc. 30th ICALP, LNCS 2719*, pages 345–356, 2003.
[30] J. I. Munro and V. Raman. Succinct representation of balanced parentheses and static trees. *SIAM J. Comput.*, 31:762–776, 2001.
[31] J. I. Munro, V. Raman, and S. S. Rao. Space efficient suffix trees. *J. of Algorithms*, 39:205–222, 2001.
[32] G. Navarro and V. Mäkinen. Compressed full-text indexes. *ACM Computing Surveys*, 39(1):2:1–2:61, 2007.
[33] D. Okanohara and K. Sadakane. Practical entropy-compressed rank/select dictionary. In *ALENEX*. SIAM, 2007.
[34] R. Pagh. Low redundancy in static dictionaries with constant query time. *SIAM J. Computing*, 31:353–363, 2001.
[35] M. Pătraşcu. Succincter. In *To Appear in Proc. 49th IEEE FOCS*, 2008.
[36] M. Pătraşcu and M. Thorup. Time-space trade-offs for predecessor search. In *Proc. 38th ACM STOC*, pages 232–240, 2006.
[37] R. Raman, V. Raman, and S. S. Rao. Succinct indexable dictionaries, with applications to representing k-ary trees, prefix sums and multisets. *ACM Transactions on Algorithms*, 3(4), 2007.
[38] P. van Emde Boas, R. Kaas, and E. Zijlstra. Design and implementation of an efficient priority queue. *Mathematical Systems Theory*, 10:99–127, 1977.
[39] S. Vigna. Broadword implementation of rank/select queries. In *Proc. 7th WEA*, pages 154–168, 2008.